

# K-THEORY OF BISINGULAR PSEUDODIFFERENTIAL ALGEBRAS

KARSTEN BOHLEN

ABSTRACT. In this paper we calculate the K-theory of  $C^*$ -algebras given by the norm-closures of spaces of bisingular pseudodifferential operators. We obtain results for the *global* bisingular calculus in the flat  $(\mathbb{R}^{n_1+n_2})$  case.

## 1. INTRODUCTION AND PRELIMINARIES

The bisingular pseudodifferential operators were introduced in 1975 by L. Rodino. The bisingular calculus defines pseudodifferential operators on the cartesian product  $X \times Y$  of two compact smooth manifolds  $X$  and  $Y$ . This includes for example the external product  $P_1 \sharp P_2$  of two pseudodifferential operators. Together with the multiplicative property of the Fredholm index this was a motivation for the introduction of a general bisingular calculus. See e.g. [2] and the references contained therein. Based on this, in [1], there is described a *global* version of the calculus on the product  $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$  extending the Shubin calculus.

In this note we consider the  $C^*$ -algebras of completed pseudodifferential operators obtained from the global bisingular calculus. Our main aim is to then calculate the K-theory of these algebras.

In the ordinary case of the Shubin calculus on  $\mathbb{R}^n$  the K-theory is easily calculated from an exact sequence which is induced by the principal symbol map. The bisingular calculus by contrast has two operator-valued symbols which take values in a non-commutative symbol algebra. We will now explain more precisely the difference between these two cases.

First recall the construction of the standard exact sequence of pseudodifferential operators defined on  $\mathbb{R}^n$ . Let  $G^m(\mathbb{R}^n)$  denote the classical pseudodifferential operators (operators of Shubin type) of order  $m \in \mathbb{R}$ . Denote by  $A$  the completion of  $G^0(\mathbb{R}^n)$  in the induced  $\mathcal{L}(L^2)$  norm. We obtain a  $C^*$  algebra and the completion of  $G^{-1}$  yields  $\mathcal{K}_{\mathbb{R}^n} = \mathcal{K}(L^2)$  the algebra of compact operators on  $L^2(\mathbb{R}^n)$ . Denote by  $\sigma: G^0(\mathbb{R}^n) \rightarrow C^\infty(S^{2n-1})$  the principal symbol map and  $\bar{\sigma}: A \rightarrow C(S^{2n-1})$  its continuous extension. We have the following short exact sequence

$$0 \longrightarrow \mathcal{K}_{\mathbb{R}^n} \longrightarrow A \xrightarrow{\bar{\sigma}} C(S^{2n-1}) \longrightarrow 0. \quad (1.1)$$

By comparison consider now the *global bisingular* calculus. Here we will consider algebras  $G^{m_1, m_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  of the orders  $m_1 \in \mathbb{R}$  and  $m_2 \in \mathbb{R}$  which generalize the before mentioned Shubin calculus to products  $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ . It turns out that taking completions with regard to  $\mathcal{L}(L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}))$  we find  $\overline{G^{-1, -1}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} = \mathcal{K}$  the compact operators on  $L^2$ . If we set  $\mathcal{A} := \overline{G^{0, 0}}$  we will prove the exactness of the sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{A} \xrightarrow{\bar{\sigma}_1 \oplus \bar{\sigma}_2} \Sigma \longrightarrow 0 \quad (1.2)$$

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with a non-commutative symbol-algebra  $\Sigma$  and the extended direct sum principal symbol map  $\bar{\sigma}_1 \oplus \bar{\sigma}_2$ .

The algebra  $\Sigma$  can be viewed as a  $C^*$ -algebra pullback, i.e. a restricted direct sum of  $C(S^{2n_1-1}, \overline{G^0(\mathbb{R}^{n_2})})$  and  $C(S^{2n_2-1}, \overline{G^0(\mathbb{R}^{n_1})})$ . We will then calculate the K-theory of  $\Sigma$ .

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## 2. GLOBAL BISINGULAR OPERATORS

For  $m_1, m_2 \in \mathbb{R}$  we denote by  $\Gamma^{m_1, m_2}(\mathbb{R}^{n_1+n_2})$  the symbol class which consists of  $a \in C^\infty(\mathbb{R}^{2n_1+2n_2})$  such that for all multi indices  $\alpha_i, \beta_i$  there is a constant  $C > 0$  with

$$|\partial_{x_1}^{\beta_1} \partial_{x_2}^{\beta_2} \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} a(x_1, \xi_1, x_2, \xi_2)| \leq C \langle x_1, \xi_1 \rangle^{m_1 - |\alpha_1| - |\beta_1|} \langle x_2, \xi_2 \rangle^{m_2 - |\alpha_2| - |\beta_2|}$$

where  $\langle x_i, \xi_i \rangle := (1 + |x_i|^2 + |\xi_i|^2)^{\frac{1}{2}}, i = 1, 2$ .

Furthermore we set

$$\Gamma^{-\infty, -\infty}(\mathbb{R}^{n_1+n_2}) := \bigcap_{m_1, m_2 \in \mathbb{R}} \Gamma^{m_1, m_2}(\mathbb{R}^{n_1+n_2}). \quad (2.1)$$

The operator  $P: C_c^\infty(\mathbb{R}^{n_1+n_2}) \rightarrow C^\infty(\mathbb{R}^{n_1+n_2})$  given by  $P = \text{op}(a)$  for  $a \in \Gamma^{m_1, m_2}$  is defined as follows

$$P(u)(x_1, x_2) := \frac{1}{(2\pi)^{n_1+n_2}} \int \int e^{ix_1\xi_1 + ix_2\xi_2} a(x_1, \xi_1, x_2, \xi_2) \hat{u}(\xi_1, \xi_2) d\xi_1 d\xi_2.$$

Such an operator has a continuous extension

$$P: H^{s_1, s_2}(\mathbb{R}^{n_1+n_2}) \rightarrow H^{s_1-m_1, s_2-m_2}(\mathbb{R}^{n_1+n_2}).$$

We denote by  $G^{m_1, m_2}(\mathbb{R}^{n_1+n_2})$  the algebra of bisingular operators of order  $(m_1, m_2)$  so obtained. It is instructive to mention that the class  $\Gamma^{0,0}(\mathbb{R}^{n_1+n_2})$  is contained in the Hörmander class  $S_{0,0}^0(\mathbb{R}^{n_1+n_2})$ . For the detailed calculus the reader is referred to [1]. In the bisingular calculus there are two principal symbols. The symbol maps are denoted by

$$\begin{aligned} \sigma_1^{m_1} &: G^{m_1, m_2} \rightarrow C^\infty(\text{T}^\bullet \mathbb{R}^{n_1}, G^{m_2}(\mathbb{R}^{n_2})), \\ \sigma_2^{m_2} &: G^{m_1, m_2} \rightarrow C^\infty(\text{T}^\bullet \mathbb{R}^{n_2}, G^{m_1}(\mathbb{R}^{n_1})) \end{aligned}$$

where  $\text{T}^\bullet \mathbb{R}^{n_i} := T^* \mathbb{R}^{n_i} \setminus \{0\}, i = 1, 2$ . We set  $\sigma_1 := \sigma_1^0, \sigma_2 := \sigma_2^0$ .

Here and in what follows we only consider classical operators and symbols with homogenous principal part.

Denote by  $\sigma_{\mathbb{R}^{n_1}}, \sigma_{\mathbb{R}^{n_2}}$  the principal symbol map of  $G^{m_1}(\mathbb{R}^{n_1})$  and  $G^{m_2}(\mathbb{R}^{n_2})$  respectively. Then define in each case the pointwise principal symbol maps

$$\begin{aligned} \tilde{\sigma}_{\mathbb{R}^{n_1}} &: C^\infty(\text{T}^\bullet \mathbb{R}^{n_2}, G^{m_1}(\mathbb{R}^{n_1})) \rightarrow C^\infty(\text{T}^\bullet \mathbb{R}^{n_1} \times \text{T}^\bullet \mathbb{R}^{n_2}), \\ \tilde{\sigma}_{\mathbb{R}^{n_1}}(F)(x_1, \xi_1, x_2, \xi_2) &:= \sigma_{\mathbb{R}^{n_1}}(F(x_2, \xi_2))(x_1, \xi_1), F \in C^\infty(\text{T}^\bullet \mathbb{R}^{n_2}, G^{m_1}(\mathbb{R}^{n_1})), \\ \tilde{\sigma}_{\mathbb{R}^{n_2}} &: C^\infty(\text{T}^\bullet \mathbb{R}^{n_1}, G^{m_2}(\mathbb{R}^{n_2})) \rightarrow C^\infty(\text{T}^\bullet \mathbb{R}^{n_1} \times \text{T}^\bullet \mathbb{R}^{n_2}), \\ \tilde{\sigma}_{\mathbb{R}^{n_2}}(G)(x_1, \xi_1, x_2, \xi_2) &:= \sigma_{\mathbb{R}^{n_2}}(G(x_1, \xi_1))(x_2, \xi_2), G \in C^\infty(\text{T}^\bullet \mathbb{R}^{n_1}, G^{m_2}(\mathbb{R}^{n_2})). \end{aligned}$$

We also require for each  $P \in G^{m_1, m_2}$  the condition

$$\tilde{\sigma}_{\mathbb{R}^{n_1}}(\sigma_2(P)) = \tilde{\sigma}_{\mathbb{R}^{n_2}}(\sigma_1(P)) \quad (2.2)$$

to hold.

**Definition 2.1.** Let  $\Sigma^{m_1, m_2}$  be the set of all pairs  $(F, G) \in C^\infty(\mathbf{T}^\bullet \mathbb{R}^{n_1}, G^{m_1}(\mathbb{R}^{n_2})) \oplus C^\infty(\mathbf{T}^\bullet \mathbb{R}^{n_2}, G^{m_2}(\mathbb{R}^{n_1}))$  such that

$$\begin{aligned} F(tx_1, t\xi_1) &= t^{m_1} F(x_1, \xi_1), t > 0 \\ G(ux_2, u\xi_2) &= u^{m_2} G(x_2, \xi_2), u > 0 \end{aligned}$$

and

$$\tilde{\sigma}_{\mathbb{R}^{n_2}}(F) = \tilde{\sigma}_{\mathbb{R}^{n_1}}(G).$$

Let  $\{F_1, G_1\} \in \Sigma^{m_1, m_2}, \{F_2, G_2\} \in \Sigma^{p_1, p_2}$  and define

$$\{F_2, G_2\} \circ \{F_1, G_1\} := \{F_2 \circ_2 F_1, G_2 \circ_1 G_1\} \in \Sigma^{m_1+p_1, m_2+p_2}.$$

Here

$$\begin{aligned} (F_2 \circ_2 F_1)(x_1, \xi_1) &:= F_2(x_1, \xi_1) \star_{\mathbb{R}^{n_2}} F_1(x_1, \xi_1), \\ (G_2 \circ_1 G_1)(x_2, \xi_2) &:= G_2(x_2, \xi_2) \star_{\mathbb{R}^{n_1}} G_1(x_2, \xi_2) \end{aligned}$$

where  $\star_{\mathbb{R}^{n_2}}$  denotes the operator product  $G^{m_2}(\mathbb{R}^{n_2}) \times G^{p_2}(\mathbb{R}^{n_2}) \rightarrow G^{m_2+p_2}(\mathbb{R}^{n_2})$  and for  $\circ_1$  analogously.

**Proposition 2.2.** *The following sequence is exact*

$$0 \longrightarrow G^{m_1-1, m_2-1}(\mathbb{R}^{n_1+n_2}) \longrightarrow G^{m_1, m_2}(\mathbb{R}^{n_1+n_2}) \xrightarrow{\sigma_1^{m_1} \oplus \sigma_2^{m_2}} \Sigma^{m_1, m_2} \longrightarrow 0. \quad (2.3)$$

*Proof.* 1. Let  $P = \text{op}(a), a \in \Gamma^{m_1, m_2}$  such that  $\sigma_1^{m_1}(P) = 0, \sigma_2^{m_2}(P) = 0$ . Then from Def. 1.6. [1] (cf. [2], Def. 2.1.) it follows that  $a \in \Gamma^{m_1-1, m_2}$  and also  $a \in \Gamma^{m_1, m_2-1}$ . Now we use that  $a$  is classical in the sense that we can find  $a_k$  homogenous of degree  $m_1 - k$  in  $(x_1, \xi_1) \in \mathbb{R}^{2n_1}$  for  $k = 0, \dots, N$  such that

$$a - \sum_{k=0}^N a_k \in \Gamma^{m_1-(N+1), m_2}$$

for each  $N \in \mathbb{N}$ . And  $b_k$  homogenous of degree  $m_2 - k$  for  $k = 0, \dots, N$  in  $(x_2, \xi_2) \in \mathbb{R}^{2n_2}$  such that

$$a - \sum_{k=0}^N b_k \in \Gamma^{m_1, m_2-(N+1)}$$

for each  $N \in \mathbb{N}$ .

Therefore by considering  $k = 0$  we can write

$$a = \tilde{b}_0 + \tilde{b} = \tilde{a}_0 + \tilde{a}$$

for  $\tilde{a}, \tilde{b} \in \Gamma^{m_1-1, m_2-1}$  as well as  $\tilde{b}_0 \in \Gamma^{m_1-1, m_2}$  homogenous in  $(x_2, \xi_2)$  of order  $m_2$  and  $\tilde{a}_0 \in \Gamma^{m_1, m_2-1}$  homogenous in  $(x_1, \xi_1)$  of order  $m_1$ . Then  $\tilde{a}_0 = \tilde{b}_0 + \tilde{b} - \tilde{a} \in \Gamma^{m_1-1, m_2-1}$  and hence  $\tilde{a}_0$  is also of order  $m_1 - 1$ . But this implies that  $\tilde{a}_0 = 0$  and analogously,  $\tilde{b}_0 = 0$ . It follows that  $a \in \Gamma^{m_1-1, m_2-1}$ .

Now for  $P = \text{op}(a), a \in \Gamma^{m_1-1, m_2-1}$  we must have  $\sigma_1(P) = 0, \sigma_2(P) = 0$  as  $\Gamma^{m_1-1, m_2-1} \subseteq \Gamma^{m_1-1, m_2} \cap \Gamma^{m_1, m_2-1}$ . It follows that  $\ker(\sigma_1 \oplus \sigma_2) = \ker \sigma_1 \cap \ker \sigma_2 = G^{m_1-1, m_2} \cap G^{m_1, m_2-1} = G^{m_1-1, m_2-1}$ .

2. Let  $(F, G) \in \Sigma^{m_1, m_2}$ . There are two maps

$$\text{sym}_i: G^{m_i}(\mathbb{R}^{n_i}) \rightarrow \Gamma^{m_i}(\mathbb{R}^{n_i}), i = 1, 2$$

which are right-inverse to  $\text{op}_i: \Gamma^{m_i}(\mathbb{R}^{n_i}) \rightarrow G^{m_i}(\mathbb{R}^{n_i})$ ,  $i = 1, 2$  modulo smoothing terms (and are defined by  $A \mapsto e^{-ix_i\xi_i} A e^{ix_i\xi_i}$ ). Now set  $r := \tilde{\sigma}_{\mathbb{R}^{n_2}}(F) \in C^\infty(\mathbf{T}^\bullet \mathbb{R}^{n_1} \times \mathbf{T}^\bullet \mathbb{R}^{n_2})$  and

$$p := \widetilde{\text{symb}_2} \circ F, q := \widetilde{\text{symb}_1} \circ G.$$

Where  $\widetilde{\text{symb}_i}$  are the pointwise evaluations of  $\text{symb}_i$  for  $i = 1, 2$ . Choose two smooth cut-off functions near 0 on  $\mathbb{R}^{2n_1}, \mathbb{R}^{2n_2}$  respectively:  $\chi_1, \chi_2$  and set

$$a := \chi_1 p + \chi_2 q - \chi_1 \chi_2 r \in \Gamma^{m_1, m_2}.$$

Then  $\sigma_1(\text{op}(a)) = F$  and  $\sigma_2(\text{op}(a)) = G$  (cf. [1], Def. 1.6 iii)). Hence  $\sigma_1 \oplus \sigma_2$  is surjective and the exactness of (2.3) is established.  $\square$

### 3. COMPLETED ALGEBRAS

In what follows we want to consider the completions of the bisingular operator algebras introduced in the last section. In particular, we want exact sequences as in (2.3) for the completed algebras.

All the results that follow not including the main Theorem can be generalized without non-trivial modifications to the bisingular calculus on compact smooth manifolds.

**Lemma 3.1.** i) *Completion with respect to the bounded linear operators on  $L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  yields*

$$\overline{G^{-1, -1}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} = \mathcal{K}(L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})) =: \mathcal{K}. \quad (3.1)$$

ii) *The algebra tensor product  $G^{m_1}(\mathbb{R}^{n_1}) \otimes G^{m_2}(\mathbb{R}^{n_2})$  is dense in  $G^{m_1, m_2}(\mathbb{R}^{n_1+n_2})$  with regard to the Fréchet topology.*

*Proof.* i) We repeat the standard argument for the benefit of the reader. We have  $G^{-\infty, -\infty}(\mathbb{R}^{n_1+n_2}) \subset G^{-1, -1}(\mathbb{R}^{n_1+n_2})$  and operators in  $G^{-\infty, -\infty}$  have integral kernel in  $S(\mathbb{R}^{n_1+n_2})$ . Now  $S(\mathbb{R}^{n_1+n_2})$  is dense in  $L^2(\mathbb{R}^{n_1+n_2})$ . Thus  $G^{-\infty, -\infty}$  is dense in the Hilbert-Schmidt operators  $\mathcal{L}^2(L^2(\mathbb{R}^{n_1+n_2}))$  which is in turn dense in the compacts  $\mathcal{K}$ .

ii) Note that  $\Gamma^{-\infty, -\infty}(\mathbb{R}^{2n_1+2n_2})$  is dense in  $\Gamma^{m_1, m_2}(\mathbb{R}^{n_1+n_2})$ . From the inclusions  $\Gamma^{-\infty, -\infty} \subset S_{0,0}^{-\infty} \subset S(\mathbb{R}^{2n_1+2n_2})$  it follows that  $\Gamma^{-\infty, -\infty} \subset S(\mathbb{R}^{2n_1+2n_2})$ . But  $S(\mathbb{R}^{2n_1+2n_2})$  is topologically isomorphic to  $S(\mathbb{R}^{2n_1}) \hat{\otimes} S(\mathbb{R}^{2n_2})$  where  $\hat{\otimes}$  denotes the completed projective tensor product. As  $S(\mathbb{R}^{2n_1}) \hat{\otimes} S(\mathbb{R}^{2n_2}) \subset \Gamma^{m_1}(\mathbb{R}^{n_1}) \otimes \Gamma^{m_2}(\mathbb{R}^{n_2})$  it follows that  $\Gamma^{-\infty, -\infty}$  is also contained in  $\Gamma^{m_1} \otimes \Gamma^{m_2}$ . This implies the required density.  $\square$

*Notation 3.2.* Throughout the rest of this paper we fix the following notation. Define the respective  $\mathcal{L}(L^2)$  completions

$$\begin{aligned} A_1 &:= \overline{G^0(\mathbb{R}^{n_1})}, A_2 := \overline{G^0(\mathbb{R}^{n_2})}, \mathcal{K}_1 := \overline{G^{-1}(\mathbb{R}^{n_1})}, \mathcal{K}_2 := \overline{G^{-1}(\mathbb{R}^{n_2})}, \\ A^{i,j} &:= \overline{G^{i,j}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}, \quad i, j \in \{-1, 0, 1\}. \end{aligned}$$

For  $i = j = -1$  by the Lemma we have  $A^{-1, -1} = \mathcal{K}$ , the compact operators on the Hilbert space  $L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ .

**Corollary 3.3.** *The completion of  $A_1 \otimes A_2$  with respect to any  $C^*$ -tensor norm is isomorphic to  $\mathcal{A}$ . Also the completion of  $\mathcal{K}_1 \otimes \mathcal{K}_2$  is isomorphic to  $\mathcal{K}$ .*

*Proof.* We endow the algebraic tensor product  $A_1 \otimes A_2$  with the spatial tensor product norm  $\sigma$  (c.f. [3], Def. T.5.16). Since we have two injective  $*$ -representations  $A_1 \rightarrow \mathcal{L}(L^2(\mathbb{R}^{n_1})), A_2 \rightarrow \mathcal{L}(L^2(\mathbb{R}^{n_2}))$  given by the inclusions it follows that the dense homomorphism  $A_1 \otimes A_2 \rightarrow \mathcal{A}$  (by 3.1, ii)) is an isometry from  $(A_1 \otimes A_2, \sigma) \rightarrow (\mathcal{A}, \|\cdot\|)$ . Thus  $A_1 \otimes_\sigma A_2 \cong \mathcal{A}$ . As extensions of nuclear  $C^*$ -algebras  $A_1$  and  $A_2$  are nuclear (c.f.

[3], Thm. T.6.27 and (1.1)). Therefore the isomorphism holds for any  $C^*$  norm on the tensor product. The same argument applies to the case  $\mathcal{K}_1 \otimes_\sigma \mathcal{K}_2 \cong \mathcal{K}$ .  $\square$

From now on we write for this reason  $A_1 \otimes A_2$  for the  $C^*$ -tensor product and identify  $A_1 \otimes A_2$  with  $\mathcal{A}$ . In the same way we identify  $\mathcal{K}_1 \otimes \mathcal{K}_2$  with  $\mathcal{K}$ . In view of the continuous extensions  $\sigma_{\mathbb{R}^{n_1}} : A_1 \rightarrow C(S^{2n_1-1})$ ,  $\sigma_{\mathbb{R}^{n_2}} : A_2 \rightarrow C(S^{2n_2-1})$  (c.f. (1.1)) the pointwise principal symbol maps are given by

$$\begin{aligned}\tilde{\sigma}_{\mathbb{R}^{n_1}} : C(S^{2n_1-1}, A_2) &\rightarrow C(S^{2n_1-1} \times S^{2n_2-1}), \\ \tilde{\sigma}_{\mathbb{R}^{n_2}} : C(S^{2n_2-1}, A_1) &\rightarrow C(S^{2n_1-1} \times S^{2n_2-1}).\end{aligned}$$

The completed symbol-algebra  $\Sigma$  is obtained as the restricted direct sum of  $C(S^{2n_1-1}, A_2) \oplus C(S^{2n_2-1}, A_1)$  together with condition (2.2). First we define the projections  $\pi_1, \pi_2$  from  $\Sigma^{0,0}$  onto the first, respectively second component. We denote the continuous extensions of these projections by the same symbols (for simplicity). With this  $\Sigma$  is a  $C^*$ -algebra with norm:

$$\begin{aligned}\|(F, G)\| &:= \sup\{\|F\|_1, \|G\|_2\}, (F, G) \in \Sigma, \\ \|F\|_1 &:= \sup_{(x_2, \xi_2) \in S^{2n_2-1}} \|F(x_2, \xi_2)\|_{A_1}, \\ \|G\|_2 &:= \sup_{(x_1, \xi_1) \in S^{2n_1-1}} \|G(x_1, \xi_1)\|_{A_2}.\end{aligned}$$

The pullback  $\Sigma$  can thus be written in terms of the following diagram:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\pi_1} & C(S^{2n_2-1}) \otimes A_1 \\ \pi_2 \downarrow & & \downarrow \tilde{\sigma}_{\mathbb{R}^{n_1}} \\ C(S^{2n_1-1}) \otimes A_2 & \xrightarrow{\tilde{\sigma}_{\mathbb{R}^{n_2}}} & C(S^{2n_1-1} \times S^{2n_2-1}) \end{array} \quad (3.2)$$

Then consider the two short exact sequences:

$$\begin{aligned}0 \longrightarrow \mathcal{K}_1 \otimes A_2 &\longrightarrow \mathcal{A} \xrightarrow{\sigma_{\mathbb{R}^{n_1}} \otimes \text{id}_2} C(S^{2n_1-1}) \otimes A_2 \longrightarrow 0 \\ 0 \longrightarrow \mathcal{K}_2 \otimes A_1 &\longrightarrow \mathcal{A} \xrightarrow{\text{id}_1 \otimes \sigma_{\mathbb{R}^{n_2}}} C(S^{2n_2-1}) \otimes A_1 \longrightarrow 0.\end{aligned}$$

This is already enough information to construct the pullback  $\Sigma$  (up to isomorphism) and identify it with  $\mathcal{A}/\mathcal{K}$ .

Consider the following diagram which is put together by tensoring of the standard exact sequences and application of quotient mappings:

$$\begin{array}{ccccc}
 & & C(S^{2n_1-1}, \mathcal{K}_2) & & \\
 & \nearrow & \uparrow & \searrow & \\
 \mathcal{K}_2 \otimes A_1 & & & & C(S^{2n_1-1}, A_2) \\
 \nearrow & \searrow & \nearrow & \searrow & \nearrow \\
 \mathcal{K}_1 \otimes \mathcal{K}_2 & \xrightarrow{\quad} & A_1 \otimes A_2 & \xrightarrow{\quad q \quad} & \tilde{\Sigma} \\
 \searrow & \nearrow & \searrow & \nearrow & \searrow \\
 \mathcal{K}_1 \otimes A_2 & & & & C(S^{2n_2-1}, A_1) \\
 & \searrow & \nearrow & & \\
 & & C(S^{2n_2-1}, \mathcal{K}_1) & & 
 \end{array}$$

Additional mappings and labels in the diagram:

- $\sigma_{\mathbb{R}^{n_1}} \otimes \text{id}$  from  $A_1 \otimes A_2$  to  $C(S^{2n_1-1}, A_2)$
- $\text{id} \otimes \sigma_{\mathbb{R}^{n_2}}$  from  $A_1 \otimes A_2$  to  $C(S^{2n_2-1}, A_1)$
- $\tilde{\pi}_1$  from  $C(S^{2n_1-1}, A_2)$  to  $\tilde{\Sigma}$
- $\tilde{\pi}_2$  from  $C(S^{2n_2-1}, A_1)$  to  $\tilde{\Sigma}$
- $\tilde{\sigma}_{\mathbb{R}^{n_2}}$  from  $C(S^{2n_1-1}, A_2)$  to  $C(S^{2n_1-1} \times S^{2n_2-1})$
- $\tilde{\sigma}_{\mathbb{R}^{n_1}}$  from  $C(S^{2n_2-1}, A_1)$  to  $C(S^{2n_1-1} \times S^{2n_2-1})$

Here we denote by  $\tilde{\Sigma}$  a pullback

$$C(S^{2n_1-1}, A_2) \oplus_{C(S^{2n_1-1} \times S^{2n_2-1})} C(S^{2n_2-1}, A_1).$$

The map  $q$  is well-defined as follows:

$$q(x) := (\sigma_{\mathbb{R}^{n_1}} \otimes \text{id})(x) \oplus (\text{id} \otimes \sigma_{\mathbb{R}^{n_2}})(x), x \in A_1 \otimes A_2$$

with kernel

$$\ker q = \ker(\sigma_{\mathbb{R}^{n_1}} \otimes \text{id}) \cap \ker(\text{id} \otimes \sigma_{\mathbb{R}^{n_2}}) = (\mathcal{K}_1 \otimes A_2) \cap (A_1 \otimes \mathcal{K}_2) = \mathcal{K}_1 \otimes \mathcal{K}_2.$$

Also  $q$  is surjective: Let  $(F, G) \in \tilde{\Sigma}$  and choose  $x \in A_1 \otimes A_2$  such that  $(\sigma_{\mathbb{R}^{n_1}} \otimes \text{id})(x) = F$ . Then

$$\tilde{\sigma}_{\mathbb{R}^{n_2}}((\sigma_{\mathbb{R}^{n_1}} \otimes \text{id})(x) - G) = \tilde{\sigma}_{\mathbb{R}^{n_1}}((\text{id} \otimes \sigma_{\mathbb{R}^{n_2}})(x)) - \tilde{\sigma}_{\mathbb{R}^{n_2}}(G) = \tilde{\sigma}_{\mathbb{R}^{n_1}}(F) - \tilde{\sigma}_{\mathbb{R}^{n_2}}(G) = 0$$

which implies  $(\sigma_{\mathbb{R}^{n_1}} \otimes \text{id})(x) - G \in \ker \tilde{\sigma}_{\mathbb{R}^{n_2}} = C(S^{2n_1}, \mathcal{K}_2)$ . But  $C(S^{2n_1-1}, \mathcal{K}_2) = (\sigma_{\mathbb{R}^{n_1}} \otimes \text{id})(A_1 \otimes \mathcal{K}_2)$ . Hence we find  $x_0 \in A_1 \otimes \mathcal{K}_2$  with  $(\sigma_{\mathbb{R}^{n_1}} \otimes \text{id})(x + x_0) = G$ . Hence  $q(x + x_0) = (F, G)$ .

Now the continuous extension of  $\sigma_1 \oplus \sigma_2$  is denoted by  $\sigma$ . We see that  $\sigma$  is surjective because  $(\sigma_1 \oplus \sigma_2)(P^*) = (\sigma_1 \oplus \sigma_2)(P)^*$  and  $\sigma_1 \oplus \sigma_2$  is surjective. Also  $\mathcal{K} \subset \ker \sigma$  by Prop. 2.2. Then  $q$  restricted to  $G^{0,0}$  agrees with  $\sigma_1 \oplus \sigma_2$  which implies  $\ker \sigma \subset \mathcal{K}$ .

Finally, by uniqueness of the pullback  $\Sigma$  is isomorphic to  $\tilde{\Sigma}$ .

We obtain with this the following result.

**Proposition 3.4.** *We have an isomorphism  $\mathcal{A}/\mathcal{K} \cong \Sigma$  induced by the continuous extension  $\sigma$  of the direct-sum principal symbol.*

*Remark 3.5.* As the isomorphism induced by  $\sigma_1 \oplus \sigma_2$  (continuous extensions) is automatically isometric we obtain the norm-equality

$$\inf_{K \in \mathcal{K}} \|P + K\| = \sup\{\|\sigma_1(P)\|_1, \|\sigma_2(P)\|_2\}, \quad P \in \mathcal{A}$$

which is an expected standard result for a pseudodifferential calculus.

Next we will calculate the K-theory of the completed algebras.

**Theorem 3.6.** *We have the following K-theory*

$$K_0(A^{i,j}) \cong \mathbb{Z}, K_1(A^{i,j}) \cong 0, i, j = 0, -1$$

$$K_0(\Sigma) \cong \ker(\tilde{\sigma}_{\mathbb{R}^{n_1}*} - \tilde{\sigma}_{\mathbb{R}^{n_2}*}) \cong \mathbb{Z}$$

$$K_1(\Sigma) \cong \operatorname{coker}(\tilde{\sigma}_{\mathbb{R}^{n_1}*} - \tilde{\sigma}_{\mathbb{R}^{n_2}*}) \cong \mathbb{Z}$$

where we set  $\tilde{\sigma}_{\mathbb{R}^{n_1}*} := K_0(\tilde{\sigma}_{\mathbb{R}^{n_1}})$ ,  $\tilde{\sigma}_{\mathbb{R}^{n_2}*} := K_0(\tilde{\sigma}_{\mathbb{R}^{n_2}})$  for the induced maps in K-theory.

*Proof.* *i)* We first note that  $K_0(A_1) = K_0(A_2) \cong \mathbb{Z}$ ,  $K_1(A_1) = K_1(A_2) \cong 0$ . Using that  $K_i(C(S^{2n_j-1})) = \mathbb{Z}, i, j = 0, 1$  (c.f. [3], 6.5) this follows by application of the six-term exact sequence applied to

$$0 \longrightarrow \mathcal{K}_1 \longrightarrow A_1 \xrightarrow{\sigma_{\mathbb{R}^{n_1}}} C(S^{2n_1-1}) \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{K}_2 \longrightarrow A_2 \xrightarrow{\sigma_{\mathbb{R}^{n_2}}} C(S^{2n_2-1}) \longrightarrow 0.$$

Note that the index map in K-theory in each case is surjective, e.g. by Fedosov's index formula. From the six-term exact sequence we see that the index map from  $\mathbb{Z} \rightarrow \mathbb{Z}$  is in fact an isomorphism.

Next note that  $A_1, A_2$  are separable and as extensions of nuclear  $C^*$ -algebras themselves nuclear. Also as we have just seen the K-theory groups are torsion-free. Hence we can apply Künneth's theorem ([3], p. 171) as follows for  $i, j = 0, 1$ :

$$K_i(C(S^{2n_j-1}) \otimes A_1) = K_i(C(S^{2n_j-1}) \otimes A_2) \cong \mathbb{Z},$$

$$K_0(A_1 \otimes \mathcal{K}_2) = K_0(A_2 \otimes \mathcal{K}_1) \cong \mathbb{Z},$$

$$K_1(A_1 \otimes \mathcal{K}_2) = K_1(A_2 \otimes \mathcal{K}_1) \cong 0.$$

It follows with Prop. 3.1, *ii*):

$$K_0(A^{-1,0}) = K_0(A^{0,-1}) \cong \mathbb{Z}, K_1(A^{0,-1}) = K_1(A^{-1,0}) \cong 0.$$

*ii)* We calculate the K-theory of the pullback  $\Sigma$  via Mayer-Vietoris in K-theory (cf. [3], 11.D).

This gives:

$$\begin{array}{ccccc} K_0(\Sigma) & \xrightarrow{\pi_1^* \oplus \pi_2^*} & K_0(-, A_1) \oplus K_0(-, A_2) & \xrightarrow{\tilde{\sigma}_{\mathbb{R}^{n_1}*} - \tilde{\sigma}_{\mathbb{R}^{n_2}*}} & K_0(C(S^{2n_1-1} \times S^{2n_2-1})) \\ \delta \uparrow & & & & \downarrow \epsilon \\ K_1(C(S^{2n_1-1} \times S^{2n_2-1})) & \xleftarrow{\tilde{\sigma}_{\mathbb{R}^{n_1}*} - \tilde{\sigma}_{\mathbb{R}^{n_2}*}} & K_1(-, A_1) \oplus K_1(-, A_2) & \xleftarrow{\pi_1^* \oplus \pi_2^*} & K_1(\Sigma) \end{array}$$

Here  $K_i(-, A_1) = K_i(C(S^{2n_2-1}, A_1))$ ,  $K_i(-, A_2) = K_i(C(S^{2n_1-1}, A_2))$ ,  $i = 1, 2$ .

With the isomorphisms already established in *i)* we just have to calculate the maps on the generators of  $\mathbb{Z} \oplus \mathbb{Z}$  in each case. So we denote by  $[1_1]_0, [1_2]_0$  the generators of  $K^0(S^{2n_1-1}), K^0(S^{2n_2-1})$  respectively. As well as by  $[u_1]_1, [u_2]_1$  the unitary generators of  $K^1(S^{2n_1-1}), K^1(S^{2n_2-1})$  respectively.

Then  $K_0(\tilde{\sigma}_{\mathbb{R}^{n_1}}) - K_0(\tilde{\sigma}_{\mathbb{R}^{n_2}}) = \tilde{\sigma}_{\mathbb{R}^{n_1}*} - \tilde{\sigma}_{\mathbb{R}^{n_2}*}$  is given by

$$\mathbb{Z} \oplus \mathbb{Z} \ni (k_0, l_0) \mapsto (k_0 - l_0, 0). \quad (3.3)$$

This follows since from the short exact sequence  $0 \rightarrow \mathcal{K}_1 \rightarrow A_1 \rightarrow C(S^{2n_1-1}) \rightarrow 0$  and similarly for  $A_2$ , we see that the generators of  $K_0(A_1), K_0(A_2)$  are determined by  $[1_1]_0, [1_2]_0$  respectively.

Secondly, the map  $K_1(\tilde{\sigma}_{\mathbb{R}^{n_1}}) - K_1(\tilde{\sigma}_{\mathbb{R}^{n_2}}) = \tilde{\sigma}_{\mathbb{R}^{n_1}}^* - \tilde{\sigma}_{\mathbb{R}^{n_2}}^*$  is given by

$$\mathbb{Z} \oplus \mathbb{Z} \ni (k_1, l_1) \mapsto (k_1, -l_1) \in \mathbb{Z} \oplus \mathbb{Z}. \quad (3.4)$$

To prove this observe that

$$K_1(C(S^{2n_1-1}) \otimes A_2) \cong K_1(C(S^{2n_1-1})) \otimes K_0(A_2) \cong \mathbb{Z},$$

$$K_1(C(S^{2n_2-1}) \otimes A_1) \cong K_1(C(S^{2n_2-1})) \otimes K_0(A_1) \cong \mathbb{Z}.$$

Then  $\tilde{\sigma}_{\mathbb{R}^{n_1}}^*$  maps  $[u_1]_1 \otimes [1_2]_0$  to  $([u_1]_1 \otimes [1_2]_0, 0)$  and  $\tilde{\sigma}_{\mathbb{R}^{n_2}}^*$  maps  $[u_2]_1 \otimes [1_1]_0$  to  $(0, [u_2]_1 \otimes [1_1]_0)$ .

The morphism (3.4) is an isomorphism and hence the preceding and following arrows are zero maps. With this we have:

$$\begin{array}{ccccc} K_0(\Sigma) & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{(k_0, l_0) \mapsto (k_0 - l_0, 0)} & \mathbb{Z} \oplus \mathbb{Z} \\ \uparrow 0 & & & & \downarrow \epsilon \\ \mathbb{Z} \oplus \mathbb{Z} & \xleftarrow{(k_1, -l_1) \mapsto (k_1, l_1)} & \mathbb{Z} \oplus \mathbb{Z} & \xleftarrow{0} & K_1(\Sigma) \end{array}$$

It follows

$$K_0(\Sigma) \cong \ker\{\mathbb{Z} \oplus \mathbb{Z} \ni (k_0, l_0) \mapsto (k_0 - l_0, 0) \in \mathbb{Z} \oplus \mathbb{Z}\} \cong \Delta \cong \mathbb{Z},$$

$$K_1(\Sigma) \cong \operatorname{coker}\{\mathbb{Z} \oplus \mathbb{Z} \ni (k_0, l_0) \mapsto (k_0 - l_0, 0) \in \mathbb{Z} \oplus \mathbb{Z}\} \cong \mathbb{Z}^2 / \mathbb{Z} \cong \mathbb{Z},$$

where  $\Delta$  denotes the diagonal in  $\mathbb{Z}^2$ .

Finally, we calculate the K-theory of  $\mathcal{A} \cong A_1 \otimes A_2$  by again applying Künneth's theorem. This yields  $K_0(\mathcal{A}) \cong \mathbb{Z}, K_1(\mathcal{A}) \cong 0$ .  $\square$

*Remark 3.7.* If we analyse the proof of Theorem 3.6 we can easily write down the index formula by integrating the Chern character.

Note that the exponential map  $\epsilon$  in the Mayer Vietoris sequence is onto with constant kernel. Therefore it is given by the projection  $(k, l) \mapsto l$ . Denote by  $\beta: \mathbb{Z} \rightarrow \mathbb{Z}^2$  a splitting, i.e. a right-inverse homomorphism to  $\epsilon$ . E. g. we can choose  $\beta(l) := (l \cdot m, l)$  for some fixed  $m \in \mathbb{Z}$ .

Fix a positive orientation on  $S^{2n_1-1} \times S^{2n_2-1}$ . Then define the *topological index* as follows

$$\operatorname{ind}_t(a) := \int_{S^{2n_1-1} \times S^{2n_2-1}} \operatorname{ch}([\beta(a)]_0).$$

For an operator  $P \in \mathcal{A}$  with  $\operatorname{ind}(P) = 1$  then the topological and analytical index are equal.



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<sup>1</sup> LEIBNIZ UNIVERSITY HANNOVER, GERMANY

*E-mail address:* [bohlen.karsten@math.uni-hannover.de](mailto:bohlen.karsten@math.uni-hannover.de)